A Natural Basis for Spinor and Vector Fields on the Noncommutative sphere

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Abstract

The product of two Heisenberg-Weil algebras contains the Jordan-Schwinger representation of $su(2)$. This Algebra is quotiented by the square-root of the Casimir to produce a non-associative algebra denoted by Ψ. This algebra may be viewed as the right-module over one of its associative subalgebras which corresponds to the algebra of scalar fields on the noncommutative sphere. It is now possible to interpret other subspaces as the space of spinor or vector fields on the noncommutative sphere. A natural basis of Ψ is given which may be interpreted as the deformed entries in the rotation matrices of $SU(2)$.

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1 Introduction

The noncommutative or "fuzzy" sphere has been considered by several authors in different contexts, including the general quantisation procedure, coherent states, noncommutative geometry, the theory of membranes, and the quantum Hall effect. (See references in [1, 2, 3, 4])

Normally the approximation for the algebra of functions on a sphere (scalar fields) is in terms of matrices, [2, 5, 6] where the functions on a sphere appear only in the limit as the size of the matrix tends to infinity. An alternative approach was presented in [1] as the a two parameter algebra of polynomials $\mathcal{P}(\varepsilon, R)$ with $\varepsilon, R \in \mathbb{R}$.

The standard way of defining a vector field is to consider it as a derivation on the algebra of functions. If the latter is replaced by a matrix algebra then all such derivations are the adjoint action of elements in that algebra [2, 7]. If $\{x^1, x^2, x^3\}$ are the coordinate functions for the noncommutative sphere then the corresponding derivations $X_i = \mathrm{Ad}_{x^i}$ obey the equation $\sum_{i=1}^{3} x^i X_i = 0$ only in the commutative limit. Here we give an alternative definition for the analogue of vector fields which solves this equation identically in the noncommutative case, but these vector fields are derivations only in the limit $\varepsilon = 0$.

The idea of using the Jordan-Schwinger representation of $su(2)$ to describe spinor fields have be pursued by many authors [3, 4, 8, 9, 10]. They discovered that one could view spinor fields as rectangular matrices.

This article may be consider as an extension of the [1] to spinor and vector fields by the use of the Jordan-Schwinger representation of $su(2)$.

1.1 Structure of the article

We present $\mathcal{W}$ an algebra which is the product of two Heisenberg-Weil algebra generated by $[a_-, a_+] = \varepsilon$ and $[b_-, b_+] = \varepsilon$. This naturally contains the Jordan-Schwinger representation of both $su(2)$ given by $\{J_0, J_+, J_-\}$ and $su(1,1)$ given by $\{K_0, K_+, K_-\}$ defined in (2.3).

The idea is to consider the algebra generated by $\{J_0, J_+, J_-\}$ to be equivalent to the analogue of the algebra of functions (or scalar fields) on the sphere as presented in [1]. In this paper the algebra generated by $\{J_0, J_+, J_-\}$ which is the universal covering algebra of $su(2)$ is quotiented by the ideal generated the Casimir $J_0^2 + \frac{1}{2}J_+J_- + \frac{1}{2}J_-J_+ \sim R^2$. where $R^2 \in \mathbb{R}$. Since the Casimir operator commutes with the set $\{J_0, J_+, J_-\}$ this ideal is two sided and the corresponding quotient is an algebra.

In this paper by contrast the Casimir operator is given by $J_0^2 + \frac{1}{2}J_+J_- + \frac{1}{2}J_-J_+ = K_0^2 - \frac{1}{4}\varepsilon^2$. We would like to take the square root of this equation and consider quotienting $\mathcal{W}$ by the ideal generated by $K_0 \sim \hat{R} = (R^2 + \frac{1}{4}\varepsilon^2)^{1/2}$. However although K_0 commutes with any polynomial in $\{J_0, J_+, J_-\}$, it does not commute with all the elements in $\mathcal{W}$. Therefore the left ideal generated by $K_0 \sim \hat{R}$ is not two sided and the corresponding quotient product is not associative. However our persistence is rewarded.

We define the subspace $\Psi \subset \mathcal{W}$ as all the totally symmetric polynomials in (a_+, a_-, b_+, b_-), which contain no factor of K_0. This is achieved by defining the *formal trace*. We can define two

There is a natural basis of Ψ given by the set $\{\psi_n^{(r,m)} \mid 2n, n+r, n+m \in \mathbb{Z}, n \geq 0, |r| \leq n|m| \leq n\}$. The basis element $\psi_n^{(r,m)}$ is the unique (up to scale factor) homogeneous polynomial of degree $2n$, which is an eigenstate of both Ad_{K_0} and Ad_{J_0} with corresponding eigenvalues εr and εm. These basis elements are also orthogonal with respect to a sesquilinear form defined in a similar way to the sesquilinear form in [1].

These basis elements may be thought of as entries in a deformed rotation matrix, since, in the limit $\varepsilon = 0$, $\psi_n^{(r,m)}$ is proportional to $D^n_{mr}(\alpha,\beta,\gamma)$ where α,β,γ are the Eular angles. As a result we may interpret the algebra (Ψ,ρ) as the noncommutative and nonassociative analogue of functions on the group $SU(2)$.

We label the eigenspace $\Psi_\bullet^{(r,\bullet)} = \mathrm{span}\{\psi_n^{(r,m)} \forall n, m\}$. Since the eigenspace $\Psi_\bullet^{(0,\bullet)}$ corresponds to all elements of Ψ which commute with K_0, these elements can be written as polynomials in $\{J_0, J_+, J_-\}$ and thus correspond to the scalar fields on the noncommutative sphere. Now as stated the product ρ on Ψ is not associative but we show that $\Psi_\bullet^{(r,\bullet)}$ for each r can be viewed as a right module over $\Psi_\bullet^{(0,\bullet)}$. This is analogous to the fact that standard spinor and vector fields are modules over the space of scalar fields.

It is natural to call the set $\Psi_\bullet^{(1,\bullet)}$ the analogue of vectors fields, and to call $\Psi_\bullet^{(-1,\bullet)}$ the analogue of covectors fields or 1-forms. Using this interpretation we may define an exterior "derivative" $d : \Psi_\bullet^{(0,\bullet)} \mapsto \Psi_\bullet^{(-1,\bullet)}$. This map obeys the Leibniz rule only in the limit $\varepsilon = 0$. Likewise the interpretation of vectors as derivations on the algebra of function is also true only in this limit. By contrast the equation $x^i X_i = 0$ is true for all ε not just in the limit. Thus its swings and roundabouts and depends on ones personal conviction as to which properties of a vector are fundermental and thus should be valid for all ε and which need be true only in the limit.

The spaces $\Psi_\bullet^{(1/2,\bullet)}$ and $\Psi_\bullet^{(1/2,\bullet)}$, are the analogue of spinor fields on the noncommutative sphere. They may be viewed as contra-variant and co-variant vector fields, or positive and negative chiral vector fields depending on ones interpretation. We justify the definition of a spinor field on two grounds. First the product of two spinor fields is a vector, and secondly the rotation of a spinor though 2π inverses the sign of the spinor. Alternatively writing the spinors as 2-dimensional vectors with entries in $\Psi_\bullet^{(-1/2,\bullet)} \oplus \Psi_\bullet^{(1/2,\bullet)}$, we obtain the standard definition of spinors in the limit $\varepsilon = 0$.

1.2 Contents of the article

Most of this article concerns itself with the mathematical structure necessary to define Ψ and its products. In section 2 we examine the Heisenberg-Weil algebra $\mathcal{W}$. We define the formal trace and give some of its properties. In section 3 we present the vector space Ψ and some of its subspaces. We define the non associative products ρ and $\rho^\star$, and show that when they are restricted to the space of scalar fields, they are associative. We also define the sesquilinear Hermitian product that fails to be positive definite. (Although it is positive definite in the case $\varepsilon = 0$.)

In section 4 we define the orthogonal basis for Ψ in a similar vain to that in [1]. In section 5 we give useful formulae for calculating the product ρ on Ψ. We show how to write the basis elements either as rectangular matrices or in terms of Hahn polynomials.

In section 6 we demonstrate how $\psi_n^{(r,m)}$ may be regarded as the deformed rotation matrix entry, and also how $\Psi_\bullet^{(1,\bullet)}$ can be viewed as the space of vector fields and $\Psi_\bullet^{(-1,\bullet)}$ the space of covector fields, and $\Psi_\bullet^{(1/2,\bullet)}$ as the space of spinor fields on the noncommutative sphere.

Finally in section 7 we discuss some of the problems with this interpretation. We also suggest how this methodology can be extended to scalar, spinor and vector fields on more exotic symmetric spaces such as the Einstein-DeSitter universe.

1.3 Notations

The summation convention is not used in this article.

The notational difference with [1] are that (1) the basis elements are not normalised, (2) we use ε instead of κ, and (3) since the product is not associative it is written explicitly whenever used.

$\hat{R}$ is always given in relation to R by (3.8). In most cases we do not write the dependence on R and ε explicitly unless there is room for confusion. The implicit dependencies on $\hat{R}$ and ε are given by:

	$\hat{R}$	ε
$\mathcal{W}$ as a vector space	No	No
$\mathcal{W}$ as an algebra	No	Yes
$S(s,t,u,v)$	No	No
Ψ, $\Psi_n^{(\bullet,\bullet)}$, $\Psi_\bullet^{(r,\bullet)}$, $\Psi_\bullet^{(\bullet,m)}$ as vector spaces	No	No
$\rho : \mathcal{W} \mapsto \Psi$ as a projection	Yes	No
$\rho : \Psi \times \Psi$ as a product	Yes	Yes
$\psi_n^{(r,m)}$	No	No

We shall use the following notation for elements of each set:

Space	Description	General elements	Basis
$\mathcal{W}$	Polynomials in (a_+, a_-, b_+, b_-)	$w, w_1, w_2, \ldots$	$S(s,t,u,v)$
Ψ	formally traceless symmetric polynomials	$\xi, \zeta, \ldots$	$\psi_n^{(r,m)}$
$\Psi_\bullet^{(0,\bullet)}$	analogue of functions	$f, g, \ldots$	$\psi_n^{(0,m)}$
$\Psi_\bullet^{(1,\bullet)}$	analogue of vectors fields	$X, Y, \ldots$	$\psi_n^{(1,m)}$
$\Psi_\bullet^{(-1,\bullet)}$	analogue of 1-forms	$\xi, \zeta, \ldots$	$\psi_n^{(-1,m)}$
$\Psi_\bullet^{(1/2,\bullet)}$	analogue of spinors	$\xi, \zeta, \ldots$	$\psi_n^{(1/2,m)}$

2 $\mathcal{W}$-expressions: The Free Algebra of the Product of Two Heisenberg-Weil algebras

As stated in the introduction in this section we define the algebra $\mathcal{W} = \mathcal{W}(\varepsilon)$ which is the product of two Heisenberg-Weil algebras. This naturally contains the Jordan Schwinger representation of $su(2)$ and $su(1,1)$. We define a basis for this algebra in terms totally symmetric polynomials. Since we wish quotient out by $K_0 = \hat{R}$ we need to find representative of the $\mathcal{W}$ which contain no factor of K_0. This is achieved by defining the *formal trace*. We give some basic properties of the formal trace including that it commutes with the adjoint representation of $su(2)$. Finally we show a convenient way of writing the elements of $\mathcal{W}$ as non symmetric polynomials

For $\varepsilon \in \mathbb{R}$ let

$$\mathcal{W}(\varepsilon) = \{\text{free algebra of finite polynomials in } (a_+, a_-, b_+, b_-)\}\Big/ \sim \tag{2.1}$$

where

$$[a_-, a_+] \sim \varepsilon \qquad [b_-, b_+] \sim \varepsilon \qquad [a_\pm, b_\pm] \sim 0 \tag{2.2}$$

This may be viewed as the tensor product of two copies of the Heisenberg-Weil algebra $\mathcal{W}(\varepsilon) = \mathcal{W}_A(\varepsilon) \otimes \mathcal{W}_B(\varepsilon)$ where $\mathcal{W}_A(\varepsilon)$ is the algebra of polynomials of (a_+, a_-), and likewise for $\mathcal{W}_B(\varepsilon)$.

From now on we write $\mathcal{W} = \mathcal{W}(\varepsilon)$ when there is no doubt about ε. A natural basis for $\mathcal{W}$ is given by $S(s,t,u,v)$ which is the totally symmetric homogeneous polynomial with each

Let $\mathcal{W}^n$ for $2n \in \mathbb{Z}$, $n \geq 0$ be the space of all symmetric homogeneous polynomials of degree $2n$. The basis for $\mathcal{W}^n$ is now given by $\{S(s,t,u,v) \mid s+t+u+v = 2n\}$. It is easy to show the dimension of $\mathcal{W}^n$ is given by $\dim(\mathcal{W}^n) = \frac{1}{6}(2n+1)(2n+2)(2n+3)$.

The subspace $\mathcal{W}^1$ has dimension 10. Out of these 6 elements turn out to be very important and are given special names: (written in non-symmetric by a more convenient form)

$$\begin{aligned} J_0 &= \tfrac{1}{2}(a_+a_- - b_+b_-) \qquad & J_+ &= a_+b_- \qquad & J_- &= a_-b_+ \\ K_0 &= \tfrac{1}{2}(a_+a_- + b_+b_- + \varepsilon) \qquad & K_+ &= a_+b_+ \qquad & K_- &= a_-b_- \end{aligned} \tag{2.3}$$

It is easy to show that $\{J_0, J_+, J_-\}$ forms a representation of $su(2)$ with $[J_0, J_+] = \varepsilon J_+$ etc, and that $\{K_0, K_+, K_-\}$ form a representation of $su(1,1)$ with $[K_0, K_+] = \varepsilon K_+$. The Casimir of these are given by

$$J_0^2 + \tfrac{1}{2}J_+J_- + \tfrac{1}{2}J_-J_+ = K_0^2 - \tfrac{1}{4}\varepsilon^2 \tag{2.4}$$

$$K_0^2 - \tfrac{1}{2}K_+K_- - \tfrac{1}{2}K_-K_+ = J_0^2 - \tfrac{1}{4}\varepsilon^2 \tag{2.5}$$

Furthermore K_0 commutes with $\{J_0, J_+, J_-\}$ and J_0 commutes with $\{K_0, K_+, K_-\}$. The entire set $\mathcal{W}^1$ forms a representation of $so(3,2)$ [11].

There is a *conjugation* of elements in $\mathcal{W}$ given by

$$\begin{aligned} &\dagger : \mathcal{W} \mapsto \mathcal{W} \\ &(w_1w_2)^\dagger = w_2^\dagger w_1^\dagger \qquad (a_\pm)^\dagger = a_\mp \qquad (b_\pm)^\dagger = b_\mp \qquad \lambda^\dagger = \overline{\lambda} \qquad \forall w_1, w_2 \in \mathcal{W}, \lambda \in \mathbb{C} \end{aligned} \tag{2.6}$$

On the basis elements it is easy to show that

$$S(s,t,u,v)^\dagger = S(t,s,v,u) \tag{2.7}$$

We define the *formal trace* of an element $w \in \mathcal{W}$ as follows:

$$\mathrm{tr} : \mathcal{W} \mapsto \mathcal{W}, \quad : \mathcal{W}^n \mapsto \mathcal{W}^{n-1} \tag{2.8}$$

Write w as a totally symmetric polynomial. There are sixteen possible combinations for the last two elements of each term. Collecting these terms together, we can now write

$$w = w_1a_+a_- + w_2a_-a_+ + w_3b_+b_- + w_4b_-b_+ + w_5a_+^2 + \cdots + w_{16}b_-^2$$

Then

$$\mathrm{tr}(w) = w_1 + w_2 + w_3 + w_4 \tag{2.9}$$

The *adjoint* is given by $\mathrm{Ad}_T(w) = [T, w]$ for $T, w \in \mathcal{W}$. This vanishes if $\varepsilon = 0$. However the limit $\lim_{\varepsilon\to 0} \frac{1}{\varepsilon}\mathrm{Ad}_T(w)$ is defined. If T is in the set $\{J_0, J_+, J_-, K_0\}$ then we can write $T = \sum_{\mu,\nu=0}^2 \chi_+^\mu T^{\mu\nu}\chi_-^\nu$. Where $\chi_\pm^1 = a_\pm$ and $\chi_\pm^2 = b_\pm$. In this case, which covers most situations, we have

$$\lim_{\varepsilon\to 0} \tfrac{1}{\varepsilon}\mathrm{Ad}_T(w) = \sum_{\mu,\nu=0}^{2} \left(\chi_+^\mu T^{\mu\nu} \frac{\partial w}{\chi_-^\nu} - \frac{\partial w}{\chi_+^\mu} T^{\mu\nu} \chi_-^\nu \right) \tag{2.10}$$

Lemma 1. *The formal trace of a basis element is given by*

$$\operatorname{tr}(S(s,t,u,v)) = 2S(s-1,t-1,u,v) + 2S(s,t,u-1,v-1) \tag{2.11}$$

This commutes with the adjoint action of J_0, J_+, J_- and K_0

$$\operatorname{tr}\circ \operatorname{Ad}_{J_0} = \operatorname{Ad}_{J_0}\circ \operatorname{tr} \quad \operatorname{tr}\circ \operatorname{Ad}_{J_+} = \operatorname{Ad}_{J_+}\circ \operatorname{tr} \quad \operatorname{tr}\circ \operatorname{Ad}_{J_-} = \operatorname{Ad}_{J_-}\circ \operatorname{tr} \quad \operatorname{tr}\circ \operatorname{Ad}_{K_0} = \operatorname{Ad}_{K_0}\circ \operatorname{tr} \tag{2.12}$$

but not with K_+, K_-. It also commutes with taking the conjugate $\operatorname{tr}(w^\dagger) = (\operatorname{tr}(w))^\dagger$

Proof. By reasoning similar to the appendix in [1], we can split the basis element for any $d \in \mathbb{Z}^+$

$$S(u_1,u_2,u_3,u_4) = \sum_{v_1+v_2+v_3+v_4=d} S(u_1-v_1,u_2-v_2,u_3-v_3,u_4-v_4)S(v_1,v_2,v_3,v_4)$$

Setting $d=2$ then

$$\operatorname{tr}(S(s,t,u,v)) = \widetilde{\operatorname{tr}}(S(s-1,t-1,u,v)S(1,1,0,0)) + \widetilde{\operatorname{tr}}(S(s,t,u-1,v-1)S(0,0,1,1))$$

Since all other terms vanish hence (2.11).

For (2.12) we need to set up a basis for each $\mathcal{W}_A$ and $\mathcal{W}_B$ given by $S_A(s,t)$ the totally symmetric homogeneous polynomial with each term a permutation of $a_-^s\, a_+^t$ and having coefficient 1. Likewise for $S_B(u,v)$. These are related to $S(s,t,u,v)$ by

$$(s+t)!(u+v)!S(s,t,u,v) = (s+t+u+v)!S_A(s,t)S_B(u,v)$$

The commutator and anti-commutator of $S_A(s,t)$ with $a_\pm$ is given by

$$\begin{aligned} [a_+, S_A(s,t)] &= -\varepsilon(s+t)S_A(s,t-1) & [a_-, S_A(s,t)] &= \varepsilon(s+t)S_A(s-1,t) \\ [a_+, S_A(s,t)]_+ &= 2s(s+t+1)^{-1}S_A(s,t-1) & [a_-, S_A(s,t)]_+ &= 2t(s+t+1)^{-1}S_A(s-1,t) \end{aligned}$$

Where $[w_1,w_2]_+ = w_1w_2 + w_2w_1$ is the anti-commutator. The expressions for the commutators may be proved by induction on the degree $(s+t)$. The expressions for the anti-commutators by use of the formula

$$[a_+a_-, S_A(s,t)] = \varepsilon(s+t)S_A(s,t)$$

Now (2.12) follow from direct calculation. □

Lemma 2. *If $w \in \mathcal{W}$ then it may be written as a sum of elements*

$$w = \sum_{r,m} w_{rm} \qquad \text{where} \qquad \operatorname{Ad}_{K_0} w = \varepsilon r w\,,\ \operatorname{Ad}_{J_0} w = \varepsilon m w \tag{2.13}$$

and we can write w_{rm} as a non symmetric polynomial

$$w_{rm} = a_\pm^{r+m} b_\pm^{r-m} p_{rm}(J_0, K_0) \tag{2.14}$$

where

$$a_\pm^r = \begin{cases} a_+^r & \text{if } r>0 \\ 1 & \text{if } r=0 \\ a_-^{-r} & \text{if } r<0 \end{cases} \tag{2.15}$$

and likewise for $b_\pm^r$, and where $p_{rm}(J_0,K_0)$ is a polynomial in J_0 and K_0.

Proof. Since Ad_{K_0} and Ad_{J_0} commute we can diagonalise w with respect to these operators.

Since the $a_\pm$ commute with the $b_\pm$ collect all the $a_\pm$'s together. Use $a_-a_+ = J_0 + K_0 + \varepsilon/2$ and to remove a_-a_+ and $a_+a_- = J_0 + K_0 - \varepsilon/2$ to remove a_+a_-. By using $[J_0, a_+] = \varepsilon/2\, a_+$ etc we can move all J_0 and K_0 to the right. From (2.13) we know that for each term in w the number of a_+ minus the number of a_- is $m+r$. Thus what remains on the left is $a_\pm^{m+r}$. Similarly with

3 (Ψ,ρ): A Nonassociative and Noncommutative Algebra of Formally Traceless Symmetric Polynomials in $\mathcal{W}$

In this section we introduce the subspace Ψ of $\mathcal{W}$ of all formally traceless symmetric polynomials. We introduce two products ρ and $\rho^\star$ on Ψ neither of which are associative, and show they are well defined. In lemma 4 we show that Ψ contains a subspace $\Psi_\bullet^{(0,\bullet)}$ which is an associative algebra, analogous to the algebra of functions on a sphere, and that the spaces $\Psi_\bullet^{(r,\bullet)}$ may be viewed as modules over $\Psi_\bullet^{(0,\bullet)}$. We also show the order to which Ψ is associative.

Let $\Psi \subset \mathcal{W}$ be the subspace of all traceless symmetric polynomials in (a_+, a_-, b_+, b_-).

$$\Psi = \ker(\mathrm{tr}) \subset \mathcal{W} \tag{3.1}$$

We define the subspace

$$\Psi_n^{(\bullet,\bullet)} = \ker(\mathrm{tr}) \cap \mathcal{W}^n \qquad \text{for } 2n \in \mathbb{Z} \text{ and } n \geq 0 \tag{3.2}$$

i.e. the space of all formally traceless symmetric homogeneous polynomials of order n. Since $\Psi_n^{(\bullet,\bullet)}$ is the kernel of the restriction $\mathrm{tr} : \mathcal{W}^n \mapsto \mathcal{W}^{n-1}$, which is surjective, the dimension of $\Psi_n^{(\bullet,\bullet)}$ is $\dim(\Psi_n^{(\bullet,\bullet)}) = (2n+1)^2$. We define the projection π_n

$$\pi_n : \Psi \mapsto \bigoplus_{m=0}^{2n} \Psi_{m/2}^{(\bullet,\bullet)} \tag{3.3}$$

We may divide the space Ψ in three different ways, since for each $\xi \in \Psi$ both $\mathrm{Ad}_{J_0}(\xi), \mathrm{Ad}_{K_0}(\xi) \in \Psi$:

$$\Psi_\bullet^{(\bullet,m)} = \{\xi \in \Psi \mid \mathrm{Ad}_{J_0}\xi = \varepsilon m \xi\} \tag{3.4}$$

$$\Psi_\bullet^{(r,\bullet)} = \{\xi \in \Psi \mid \mathrm{Ad}_{K_0}\xi = \varepsilon r \xi\} \tag{3.5}$$

We will show in the next section that if $2n, n+m, n+r \in \mathbb{Z}$ and $n \geq 0$, $|m| \leq n$, $|r| \leq n$ then $\dim(\Psi_n^{(\bullet,\bullet)} \cap \Psi_\bullet^{(r,\bullet)} \cap \Psi_\bullet^{(\bullet,m)}) = 1$, otherwise it has dimension 0.

So far we have considered Ψ simply as a vector space. On this vector space we define two products. These products are given by the projections ρ_R and $\rho_R^\star$.

The intention is that $R \geq 0$ represents the radius of the sphere or the Casimir. From (2.4) we want $R^2 = K_0^2 - \frac{1}{4}\varepsilon^2$. We take the positive square root of this equation by $K_0 = \hat{R} = (R^2 + \frac{1}{4}\varepsilon^2)^{1/2}$. Since K_0 is not in the center of the algebra we must be careful about quotienting by this equation. We therefore consider two projections

$$\begin{aligned} \rho_R : \mathcal{W} \mapsto \Psi, &\qquad (\rho_R)^2 = \rho_R \\ \rho_R^\star : \mathcal{W} \mapsto \Psi, &\qquad (\rho_R^\star)^2 = \rho_R^\star \end{aligned} \tag{3.6}$$

defined by by their kernels

$$\begin{aligned} \ker(\rho_R) &= \{w\,(K_0 - \hat{R}) \mid w \in \mathcal{W}\} \\ \ker(\rho_R^\star) &= \{(K_0 - \hat{R})\,w \mid w \in \mathcal{W}\} \end{aligned} \tag{3.7}$$

where

$$\hat{R} = (R^2 + \tfrac{1}{4}\varepsilon^2)^{1/2}, \qquad R, \hat{R} \in \mathbb{R},\ R \geq 0, \hat{R} > 0 \tag{3.8}$$

From now on we shall write ρ and $\rho^\star$ when there is only one possible R. $\hat{R}$ will always be given

Lemma 3.

$$\mathcal{W} = \Psi \oplus \ker(\rho) = \Psi \oplus \ker(\rho^\star) \tag{3.9}$$

Proof. Let $w \in \mathcal{W}^n$, $n \neq 0$ then wK_0 has a component in $\mathcal{W}^{n+1}$. Furthermore this component has non zero trace, so $\mathrm{tr}(wK_0)$ has a component in $\mathcal{W}^n$. Now $\mathrm{tr}(w\hat{R})$ cannot have a component in $\mathcal{W}^n$ so $\mathrm{tr}(w(K_0 - \hat{R})) \neq 0$.

$$\Psi \cap \ker(\rho) = \{0\} \tag{3.10}$$

We show $\mathcal{W} = \Psi \oplus \ker(\rho_R)$ by dimensional argument. For this proof only let

$$\widehat{\mathcal{W}}^n = \bigoplus_{m=0}^{2n} \mathcal{W}^{m/2} \qquad\qquad \widehat{\Psi}^n = \bigoplus_{m=0}^{2n} \Psi^{(\bullet,\bullet)}_{m/2}$$

while we also let

$$\widehat{V}^n = \ker(\rho_R) \cap \widehat{\mathcal{W}}^n = \{w\,(K_0 - \hat{R}) \mid w \in \widehat{\mathcal{W}}^{n-1}\}$$

So

$$\dim(\widehat{V}^n) = \dim(\widehat{\mathcal{W}}^{n-1}) = \sum_{m=0}^{2n-2} \dim(\mathcal{W}^m)$$

Also

$$\dim(\widehat{\Psi}^n) = \sum_{m=0}^{2n} \dim \Psi^{(\bullet,\bullet)}_{m/2} = \sum_{m=0}^{2n} (\dim \mathcal{W}^{m/2} - \dim \mathcal{W}^{m/2-1})$$

Thus

$$\dim(\widehat{\Psi}^n) + \dim(\widehat{V}^n) = \dim(\widehat{\mathcal{W}}^n)$$

By intersecting (3.10) by $\widehat{\mathcal{W}}^n$ we have $\widehat{\Psi}^n \cap \widehat{V}^n = \{0\}$. Combining these last two equations gives $\widehat{\Psi}^n \oplus \widehat{V}^n = \widehat{\mathcal{W}}^n$. $\square$

For a given ε, R we define the products, also called ρ and $\rho^\star$, by:

$$\Psi \times \Psi \mapsto \Psi\,, \qquad \xi, \zeta \mapsto \rho(\xi\zeta)\,, \qquad \xi, \zeta \mapsto \rho^\star(\xi\zeta)\,, \tag{3.11}$$

Since $\ker(\rho)$ is not a two sided ideal, the product in Ψ is not associative and thus (Ψ, ρ) is not an algebra. For example $\rho(\rho(a_+a_-)a_-) = \rho(J_0 a_- + (\hat{R} - \varepsilon/2)a_-)$ while $\rho(a_+\rho(a_-a_-)) = \rho(a_+a_-a_-) = \rho(J_0 a_- + (\hat{R} - \varepsilon)a_-)$

However the restriction $(\Psi^{(0,\bullet)}_\bullet, \rho)$ is an associative algebra. This algebra is identified with the analogue of the algebra of functions given in [1]. Both Ψ and $\Psi^{(r,\bullet)}_\bullet$ for all r are modules overs $\Psi^{(0,\bullet)}_\bullet$ in the same way the space of spinors and vector fields are modules over the space of functions. For reasons that will become apparent we will call the elements of $\Psi^{(1,\bullet)}_\bullet$ vectors, the elements of $\Psi^{(-1,\bullet)}_\bullet$ forms, and the elements of $\Psi^{(1/2,\bullet)}_\bullet$ and $\Psi^{(-1/2,\bullet)}_\bullet$ spinors. We also define a sesquilinear product and quadratic map:

$$\langle \bullet, \bullet \rangle : \Psi \times \Psi \mapsto \mathbb{C} \qquad\qquad \langle \xi, \zeta \rangle = \pi_0(\rho_R(\xi^\dagger \zeta)) \tag{3.12}$$

$$\| \bullet \|^2 : \Psi \mapsto \mathbb{R} \qquad\qquad \|\zeta\|^2 = \langle \zeta, \zeta \rangle \tag{3.13}$$

As we will see this sesquilinear product is hermitian but not positive definite, and that $\|\zeta\|^2$

Lemma 4. *For a given ε and R, the product ρ when restricted to the space $\Psi_\bullet^{(0,\bullet)}$ is closed and associative making $(\Psi_\bullet^{(0,\bullet)}, \rho)$ an algebra. It is also equal the product $\rho^\star$ on $\Psi_\bullet^{(0,\bullet)}$.*

Both these algebras are equivalent to algebra $\mathcal{P}(\varepsilon, R)$ given by:

$$\mathcal{P}(\varepsilon, R) = \{\textit{Free noncommuting algebra of polynomials in } (J_+, J_-, J_0)\}\Big/ \sim \tag{3.14}$$

where $\sim$ are the relations

$$[J_0, J_+] \sim \varepsilon J_+ \qquad [J_0, J_-] \sim -\varepsilon J_- \qquad [J_+, J_-] \sim 2\varepsilon J_0 \qquad J_0^2 + \tfrac{1}{2} J_+ J_- + \tfrac{1}{2} J_- J_+ = R^2 \tag{3.15}$$

This algebra is described in detail in [1].

For general $w_1, w_2 \in \Psi$ we have $\rho(w_1\, \rho(w_2)) = \rho(w_1 w_2)$ whilst $\rho(\rho(w_1) w_2) - \rho(w_1 w_2) = O(\varepsilon)$. In terms of the algebra (Ψ, ρ) this means that for $\xi_1, \xi_2, \xi_3 \in \Psi$

$$\begin{aligned} \rho(\xi_1 \rho(\xi_2 \xi_3)) &= \rho(\xi_1 \xi_2 \xi_3) \\ \rho(\rho(\xi_1 \xi_2) \xi_3) - \rho(\xi_1 \xi_2 \xi_3) &= O(\varepsilon) \end{aligned} \tag{3.16}$$

If $\xi \in \Psi_\bullet^{(r_1,\bullet)}$ and $\zeta \in \Psi_\bullet^{(r_2,\bullet)}$ then $\xi^\dagger \in \Psi_\bullet^{(-r_1,\bullet)}$ and $\rho(\xi\zeta) \in \Psi_\bullet^{(r_2+r_1,\bullet)}$.

Both Ψ and $\Psi_\bullet^{(r,\bullet)}$ for all r are right modules overs $\Psi_\bullet^{(0,\bullet)}$, since given $\xi \in \Psi$ and $f, g \in \Psi_\bullet^{(0,\bullet)}$ then

$$\rho(\rho(\xi f) g) = \rho(\xi \rho(f g)) = \rho(\xi f g) \tag{3.17}$$

The sesquilinear product $\langle \bullet, \bullet \rangle$ is Hermitian

$$\langle \xi, \zeta \rangle = \overline{\langle \zeta, \xi \rangle} \qquad \forall \xi, \zeta \in \Psi \tag{3.18}$$

Proof. If $f, g \in \Psi_\bullet^{(0,\bullet)}$ then it is clear that $\mathrm{Ad}_{K_0}(fg) = 0$, so $\rho(fg) = \rho^\star(fg)$. From lemma 2 we can write $fg = a_\pm^m b_\pm^{-m} p(J_0, K_0)$, thus $\rho(fg) = \sum_m a_\pm^m b_\pm^{-m} p(J_0, \hat{R})$. But $a_\pm^m b_\pm^{-m} = J_+^m$ for $m > 0$ and $a_\pm^m b_\pm^{-m} = J_-^{-m}$ for $m < 0$ so $\rho(fg)$ is a polynomial in $\{J_0, J_+, J_-\}$ and all such polynomial are elements in $\mathcal{P}(\varepsilon, R)$. From (2.4) these two algebras are equivalent.

Given $w_1, w_2 \in \mathcal{W}$ then from (3.9) we may write uniquely $w_1 = \rho(w_1) + w_1'(K_0 - \hat{R})$. Thus $\rho(w_2 w_1) = \rho(w_2 \rho(w_1) + w_2 w_1'(K_0 - \hat{R})) = \rho(w_2 \rho(w_1))$. Whilst $\rho(\rho(w_1) w_2) - \rho(w_1 w_2) = \rho(w_1'(K_0 - \hat{R}) w_2) = \rho(w_1' \mathrm{Ad}_{K_0}(w - 2)) = O(\varepsilon)$

If $f, g \in \Psi_\bullet^{(0,\bullet)}$ and $\xi \in \Psi$ then we write $\xi f = \rho(\xi f) + \xi_1(K_0 - \hat{R})$ so $\rho(\xi f g) - \rho(\rho(\xi f) g) = \rho(\xi_1 (K_0 - \hat{R}) g) = 0$ since $[K_0, g] = 0$.

Given $\xi \in \Psi_\bullet^{(r_1,\bullet)}$ and $\zeta \in \Psi_\bullet^{(r_2,\bullet)}$ then $\langle \xi, \zeta \rangle \neq 0$ only if $r_1 = r_2$. In this case $\rho^\star(\xi^\dagger \zeta) = \rho(\xi^\dagger \zeta)$, thus $\overline{\pi_0(\rho(\zeta^\dagger \xi))} = (\pi_0(\rho(\zeta^\dagger \xi)))^\dagger = \pi_0(\rho^\star(\xi^\dagger \zeta)) = \pi_0(\rho(\xi^\dagger \zeta))$

$\square$

4 $\psi_n^{(r,m)}$ Orthogonal basis for Ψ

Theorem 5. *There is a natural basis of Ψ given by*

$$\{\psi_n^{(r,m)} \mid 2n, r+n, m+n \in \mathbb{Z},\ |r| \le n, |m| \le n\} \tag{4.1}$$

where

$$\psi_n^{(r,m)} = \varepsilon^{m-n} \left(\frac{(n+m)!}{\cdots} \right)^{1/2} (\mathrm{Ad}_{\cdots})^{n-m} (a^{n+r} b^{n-r}) \tag{4.2}$$

This definition is extended to the case $\varepsilon = 0$ by the use of (2.10). *When written as formally traceless symmetric homogeneous polynomials, $\psi_n^{(r,m)}$ is independent of R and ε. The basis elements are orthogonal with respect to the sesquilinear form $\langle \bullet, \bullet \rangle$.*

$$\langle \psi_{n_1}^{(r_1,m_1)}, \psi_{n_2}^{(r_2,m_2)} \rangle = \delta_{n_1,n_2} \delta_{r_1,r_2} \delta_{m_1,m_2} \| \psi_{n_1}^{(r_1,m_1)} \|^2 \tag{4.3}$$

where $\|\psi_n^{(r,m)}\|^2$ may be positive negative or zero and is independent of m.

For each n, the set $\{\psi_n^{(r,m)} \mid \forall r, m\}$ form an orthogonal basis for the set Ψ^n. These elements are Eigenstates of the operators Ad_{J_0}, Ad_{K_0}, and Δ.

$$\mathrm{Ad}_{J_0} \psi_n^{(r,m)} = \varepsilon m \psi_n^{(r,m)} \tag{4.4}$$

$$\Delta \psi_n^{(r,m)} = \varepsilon^2 n(n+1) \psi_n^{(r,m)} \tag{4.5}$$

$$\mathrm{Ad}_{K_0} \psi_n^{(r,m)} = \varepsilon r \psi_n^{(r,m)} \tag{4.6}$$

where $\Delta = \mathrm{Ad}_{J_0}\mathrm{Ad}_{J_0} + \frac{1}{2}\mathrm{Ad}_{J_+}\mathrm{Ad}_{J_-} + \frac{1}{2}\mathrm{Ad}_{J_-}\mathrm{Ad}_{J_+}$, so

$$\Psi_n^{(\bullet,\bullet)} \cap \Psi_\bullet^{(r,\bullet)} \cap \Psi_\bullet^{(\bullet,m)} = \mathrm{span}\{\psi_n^{(r,m)}\} \tag{4.7}$$

The Operators Ad_{J_+}, and Ad_{J_-} act as Ladder operators within $\Psi_n^{(\bullet,\bullet)} \cap \Psi_\bullet^{(r,\bullet)}$

$$\mathrm{Ad}_{J_+} \psi_n^{(r,m)} = \varepsilon (n-m)^{1/2} (n+m+1)^{1/2} \psi_n^{(r,m+1)} \tag{4.8}$$

$$\mathrm{Ad}_{J_-} \psi_n^{(r,m)} = \varepsilon (n+m)^{1/2} (n-m+1)^{1/2} \psi_n^{(r,m-1)} \tag{4.9}$$

Proof. It is clear that $\psi_n^{(r,n)}$ is a formally traceless symmetric homogeneous polynomial or degree $2n$, from lemma 1 so are $\psi_n^{(r,m)}$ for all m. Equations (4.4-4.9) may be proved in the same way as the equivalent theorem in [1, theorem 2]. Likewise for (4.3) where it is necessary to first show that $\pi_0(\rho(\mathrm{Ad}_{J_+} x)) = \pi_0(\mathrm{Ad}_{J_+}(\rho(w))) = 0$ for all $w \in \mathcal{W}$. This also comes from lemma 1. $\square$

A formula for the conjugate of the basis element $(\psi_n^{(r,m)})^\dagger$ is given in corollary 10. As stated in lemma 4, the subspace $\Psi_\bullet^{(0,\bullet)}$ is equivalent to the algebra $\mathcal{P}(\varepsilon, R)$ given in [1]. By comparing the two definitions for the basis vectors one sees that $\psi_n^{(0,m)} = P_n^m / \alpha_n$. As a point of notation, the basis vectors here are left unnormalised, by contrast to the notation used in the above article. This avoids having to continually divide by a normalisation constant, which here would depend on both r and n.

5 Useful Formulae for the Product ρ on Ψ

In this section we give a number of formulae which are useful for various calculations and the physical interpretations. All these formulae are extensions of the corresponding formulae for the basis of scalars on the noncommutative sphere [1], but with added care taken due to the nonassociative nature of Ψ.

We start with a Hilbert space representation of $\mathcal{W}$ (subsection 5.1). As well as being used to calculate the value of norm of the basis elements (subsection 5.2), it is also useful since it can be programmed into a symbolic mathematics programme such as Maple, and hence used to calculate explicit formulae. This is followed by expressions to write $\psi_n^{(r,m)}$ as rectangular matrices (subsection 5.3) and in terms of Hahn polynomials (subsection 5.4). Finally in subsection 5.5 we extend [1, Lemma 9] to give a formula which is useful for the definition of the exterior derivative.

5.1 A Hilbert space representation of $\mathcal{W}$

The $\mathcal{H}$ be a Hilbert space with basis given by $|k,j\rangle$ where $2k, j+k \in \mathbb{Z}$ and $k \geq 0$, $|j| \leq k$. The dual basis is represented by $\langle k,j|$ where $\langle k_1,j_1|k_2,j_2\rangle = \delta_{k_1k_2}\delta_{j_1j_2}$

The action of $\mathcal{W}$ on $\mathcal{H}$ is given by

$$\begin{aligned} a_+|k,j\rangle &= \varepsilon^{1/2}(k+j+1)^{1/2}|k+\tfrac{1}{2}, j+\tfrac{1}{2}\rangle & \quad b_+|k,j\rangle &= \varepsilon^{1/2}(k-j+1)^{1/2}|k+\tfrac{1}{2}, j-\tfrac{1}{2}\rangle \\ a_-|k,j\rangle &= \varepsilon^{1/2}(k+j)^{1/2}|k-\tfrac{1}{2}, j-\tfrac{1}{2}\rangle & \quad b_-|k,j\rangle &= \varepsilon^{1/2}(k-j)^{1/2}|k-\tfrac{1}{2}, j+\tfrac{1}{2}\rangle \end{aligned} \tag{5.1}$$

The dual action is given by $(\langle k_1,j_1|w)|k_2,j_2\rangle = \langle k_1,j_1|(w|k_2,j_2\rangle)$ The effect of the six elements Js and Ks are given by

$$\begin{aligned} J_0|k,j\rangle &= \varepsilon j|k,j\rangle & K_0|k,j\rangle &= \varepsilon(k+\tfrac{1}{2})|k,j\rangle \\ J_+|k,j\rangle &= \varepsilon(k-j)^{1/2}(k+j+1)^{1/2}|k,j+1\rangle & K_+|k,j\rangle &= \varepsilon(k-j+1)^{1/2}(k+j+1)^{1/2}|k+1,j\rangle \\ J_-|k,j\rangle &= \varepsilon(k+j)^{1/2}(k-j+1)^{1/2}|k,j-1\rangle & K_-|k,j\rangle &= \varepsilon(k-j)^{1/2}(k+j)^{1/2}|k-1,j\rangle \end{aligned} \tag{5.2}$$

One sees that the the effect of J_+, J_-, J_0 are exactly the same as the elements in the algebra in [1]

Lemma 6. *Given $w \in \mathcal{W}$ which is independent of R and ε,*

$$w|k,j\rangle = 0 \quad \forall |k,j\rangle \in \mathcal{H} \qquad \Longleftrightarrow \qquad \rho_R(w) = 0 \quad \forall R \qquad \Longleftrightarrow \qquad w = 0 \tag{5.3}$$

If $\mathrm{Ad}_{K_0} w = \varepsilon r w$ and $\mathrm{Ad}_{J_0} w = \varepsilon m w$ then

$$w|k,j\rangle \in \mathrm{span}\{|k+r, j+m\rangle\} \tag{5.4}$$

If $R^2 = \varepsilon^2 k(k+1)$ then

$$\rho_R(w)|k,j\rangle = w|k,j\rangle \tag{5.5}$$

$$\pi_0(\rho_R(w)) = \frac{1}{2k+1}\sum_{j=-k}^{k} \langle k,j|w|k,j\rangle \tag{5.6}$$

Proof. Given $w \in \mathcal{W}$ such that $\mathrm{Ad}_{K_0} w = \varepsilon r x$ and $\mathrm{Ad}_{J_0} w = \varepsilon m x$ we use lemma 2 to write w in form (2.13). From (5.1), (5.4) is obvious.

If $\rho_R(w) = 0 \quad \forall R$ then it is clear that $w|k,j\rangle = 0 \quad \forall j,k$. To prove the reverse $w \in \mathcal{W}$ can be written as a sum of elements $w = \sum_{rm} w_{rm}$ with w_{rm} in the eigenspace of Ad_{K_0} and Ad_{J_0}. From (5.4) each w_{rm} satisfies $w_{rm}|k,j\rangle = 0$

Using lemma 2 write

$$w_{rm} = a_\pm^{r+m} b_\pm^{r-m} p(K_0, J_0)$$

Then

$$0 = w_{rm}|k,j\rangle = p(\varepsilon(k+\tfrac{1}{2}), \varepsilon j) a_\pm^{r+m} b_\pm^{r-m}|j,k\rangle \qquad \forall j,k$$

writing $\widehat{p}(k,j) = p(\varepsilon(k+\frac{1}{2}), \varepsilon j)$ then $\widehat{p}(k,j) = 0$ for all j,k such that $2j, 2k \in \mathbb{Z}$, $k \geq \max(0,-r)$ and $\max(-k, -k-m) \leq j \leq \min(k, k-m)$. Now writing $\widehat{p}(k,j) = \sum_s p_s(j) k^s$ then $p_s(j) = 0$ for above range of j. Chosing k greater than the degree of $\widehat{p}$ implies $p_s \equiv 0$. Thus $\widehat{p} \equiv 0$, thus $w = 0$

Finally if $w \in \Psi$ writing $w = \sum_{rm} w_{rm}$ as before, from (5.4) it is clear that the right hand side of (5.6) vanishes for all but the w_{00} term. Now w_{00} may be written as a polynomial in J_0. We can use the result of [1, Theorem 3]

5.2 The value of $\|\psi_n^{(r,m)}\|^2$

Theorem 7. *The value of $\|\psi_n^{(r,m)}\|^2$ is given by*

$$\|\psi_n^{(r,m)}\|^2 = \pi_0(a_-^{n+r}a_+^{n+r}b_+^{n-r}b_-^{n-r}) = \frac{(n+r)!(n-r)!}{(2n+1)!}\prod_{s=1}^{n-r}(2\hat{R}-\varepsilon s)\prod_{s=1}^{n+r}(2\hat{R}+\varepsilon s) \tag{5.7}$$

The sign of this now depends on ε and R and is given by

$$\operatorname{sign}(\|\psi_n^{(r,m)}\|^2) = \begin{cases} 1 & \text{if } 2\hat{R}/\varepsilon > n-r \\ 0 & \text{if } 2\hat{R}/\varepsilon \in \mathbb{Z},\ 1 \le 2\hat{R}/\varepsilon \le n-r \\ (-1)^{(n-r-\lfloor 2\hat{R}/\varepsilon \rfloor)} & \text{if } 2\hat{R}/\varepsilon \notin \mathbb{Z},\ 0 < 2\hat{R}/\varepsilon < n-r \end{cases} \tag{5.8}$$

where $\lfloor 2R/\varepsilon \rfloor$ is the largest integer less than or equal to $2R/\varepsilon$.

Proof. This proof is similar to the proof of [1, Theorem 6]. Consider $\hat{R} = \varepsilon(k+\frac{1}{2})$, we have

$$\begin{aligned}
\pi_0(a_-^{n+r}a_+^{n+r}b_+^{n-r}b_-^{n-r}) &= \frac{1}{2k+1}\sum_{j=-k}^{k}\langle k,j|a_-^{n+r}a_+^{n+r}b_+^{n-r}b_-^{n-r}|k,j\rangle \\
&= \frac{\varepsilon^{2k}}{2k+1}\sum_{j=-k}^{k}\frac{(k+j+r+n)!(k-j)!}{(k+j)!(k-j-n+r)!}
\end{aligned}$$

Substituting $j = s-k$ and removing vanishing terms terms

$$\begin{aligned}
&= \frac{\varepsilon^{2k}}{2k+1}\sum_{j=-k}^{2k-n+r}\frac{(s+r+n)!(2k-s)!}{s!(2k-n+r-s)!} \\
&= \varepsilon^{2k}\frac{(r+n)!(2k)!}{(2k+1)(2k-n+r)!}\sum_{j=-k}^{2k-n+r}\frac{(r+n+1)_s(-2k+n-r)_s}{s!(-2k)_s} \\
&= \varepsilon^{2k}\frac{(r+n)!(2k)!}{(2k+1)(2k-n+r)!}F(r+n+1,n-r-2k;-2k;1) \\
&= \varepsilon^{2k}\frac{(r+n)!(2k)!}{(2k+1)(2k-n+r)!}\lim_{\kappa\to 0}F(r+n+1,n-r-2k;\kappa-2k;1) \\
&= \varepsilon^{2k}\frac{(r+n)!(2k)!}{(2k+1)(2k-n+r)!}\lim_{\kappa\to 0}\frac{\Gamma(\kappa-2k)\Gamma(\kappa-2n-1)}{\Gamma(\kappa-2k-r-n-1)\Gamma(\kappa-n+r)} \\
&= \varepsilon^{2k}\frac{(r+n)!(n-r)!(2k+1+r+n)!}{(2k+1)(2n+1)!(2k-n+r)!}
\end{aligned}$$

Substituting $\hat{R} = \varepsilon(k+\frac{1}{2})$ and using lemma 6 gives result. □

5.3 Matrix representations for $\psi_n^{(r,m)}$

If $\hat{R} = \varepsilon(k+\frac{1}{2})$ then as already noted we can quotient the space of functions $\Psi_\bullet^{(0,\bullet)}$ to give the algebra of matrices $M_{2k+1}(\mathbb{C})$. This interpretation may be extended for each $\Psi_\bullet^{(r,\bullet)}$ but with caution to reflect its non associative nature. The elements of $\Psi_\bullet^{(r,\bullet)}$ now become rectangular matrices with $2k+2r+1$ rows and $2k+1$ columns. I.e. elements of $M_{(2k+2r+1)\times(2k+1)}(\mathbb{C})$.

Lemma 8. *Given $\hat{R} = \varepsilon(k+\frac{1}{2})$ then the mapping*

$$\begin{aligned} &\varphi_k^r : \Psi_\bullet^{(r,\bullet)} \mapsto M_{(2k+2r+1)\times(2k+1)}(\mathbb{C}) \\ &(\varphi_k^r(\xi))_{\mu\nu} = \langle k+r, \mu-k-r-1|\xi|k, \nu-k-1\rangle \end{aligned} \tag{5.9}$$

where $1 \le \mu \le 2k+2r+1$ and $1 \le \nu \le 2k+1$, satisfies the following properties:

$$\varphi_k^r(\rho(\xi f)) = \varphi_k^r(\xi)\varphi_k^0(f)\,, \qquad \forall \xi \in \Psi_\bullet^{(r,\bullet)}\,, f \in \Psi_\bullet^{(0,\bullet)} \tag{5.10}$$

$$(\varphi_k^r(\xi))^\dagger \varphi_k^r(\zeta) = \varphi_k^0(\rho(\xi^\dagger\zeta)) \tag{5.11}$$

$$\varphi_k^r(\psi_n^{(r,m)}) = 0 \qquad \text{if } n \ge 2k+r+1 \tag{5.12}$$

Proof. Equation (5.10) follows from the use of (5.5) to remove the ρ, simple matrix multiplication and observing that $\sum_{\mu=1}^{2k+1} |k, \mu-k-1\rangle\langle k, \mu-k-1|$ is the unit matrix in $M_{2k+1}(\mathbb{C})$. Likewise (5.11) follows from the same plus the recognition that $\langle k, \mu-k-1|\xi^\dagger|k+r, \nu-k-r-1\rangle = ((\varphi_k^r(\xi))^\dagger)_{\mu\nu}$

For $n \ge 2k+r+1$ then from theorem 7

$$\begin{aligned} 0 &= \langle \psi_n^{(r,m)}, \psi_n^{(r,m)}\rangle = \pi_0(\rho(\psi_n^{(r,m)\dagger}\psi_n^{(r,m)})) \\ &= \tfrac{1}{2k+1}\mathrm{TR}(\varphi_k^0(\rho(\xi^\dagger\xi))) = \tfrac{1}{2k+1}\sum_{\mu=0}^{2k+2r+1}\sum_{\nu=0}^{2k+1} |\varphi_k^r(\psi_n^{(r,m)})_{\mu\nu}|^2 \end{aligned}$$

where TR represents the matrix trace. Hence (5.12) □

By counting dimensions we see that the $\ker(\varphi_k^r) = \mathrm{span}\{\psi_n^{(r,m)} \mid \forall m, n \text{ s.t. } n \ge 2k+r+1\}$. This is not an ideal in general but it does have the property: if $\xi \in \ker(\varphi_k^r)$ and $f \in \Psi_\bullet^{(0,\bullet)}$ then $\rho(\xi f) \in \ker(\varphi_k^r)$.

5.4 Writing $\psi_n^{(r,m)}$ in terms of Hahn polynomials

Lemma 9. *The highest degree term in the expansions of $\psi_n^{(r,m)}$ is given by*

$$\psi_n^{(r,m)} = (-1)^{n-\max(r,m)} \binom{2n}{n+m}^{1/2} a_\pm^{r+m}\, b_\pm^{r-m}\, J_0^{n-\max(|r|,|m|)} \;+\; \text{Lower Degree Terms in } J_0 \tag{5.13}$$

where $a_\pm$ and $b_\pm$ are given by (2.15)

Proof. By repeated application of the formula

$$\mathrm{Ad}_{J_-}(a_+^x b_+^{n+r-x} a_-^{n-r-v} b_-^v) = \varepsilon x a_+^{x-1} b_+^{n-r-x+1} a_-^{n-r-v} b_-^v - \varepsilon v a_+^x b_+^{n+r-x} a_-^{n-r-v+1} b_-^{v-1} \tag{5.14}$$

we see that

$$\begin{aligned} &(\mathrm{Ad}_{J_-})^{n-m}(a_+^{n+r} b_+^{n-r}) \\ &\quad = \varepsilon^{n-m} \sum_{s=\max(0,r-m)}^{\min(n-m,n+r)} \frac{(n-m)!(n+r)!(n-r)!(-1)^{n-m-s}}{s!(n-m-s)!(n+r-s)!(m-r+s)!} a_+^{n+r-s} b_+^s a_-^{n-m-s} b_-^{m-r+s} \end{aligned} \tag{5.15}$$

Using $a_+a_- = J_0 + \mathrm{LDT}$ and $b_+b_- = -J_0 + \mathrm{LDT}$, where LDT means "Lower Degree Terms in J_0", we have

$$a_+^{n+r-s} b_+^s a_-^{n-m-s} b_-^{m-r+s} = a_\pm^{r+m} b_\pm^{r-m} J_0^{n+\min(r,-m)+\min(0,m-r)}(-1)^{s+\min(0,m-r)} + \mathrm{LDT}$$

Where the last expression comes from considering each of the four case below separately.

We have to consider separately the cases $m \geq r$ and $r \geq m$. We use the equations $(N+s)! = N!(N+1)_s$ and $N! = (N-s)!(-1)^s(-N)_s$ to convert our sum into a hypergeometric function. Since the coefficient is 1 we can use the summation formula to get an explicit form in terms of factorials.

For $m \geq r$ we have

$$
\begin{aligned}
&(\mathrm{Ad}_{J_-})^{n-m}(a_+^{n+r} b_+^{n-r}) \\
&= \varepsilon^{n-m} a_\pm^{r+m} b_\pm^{r-m} J_0^{n-\max(|r|,|m|)} \frac{(-1)^{n-m}(n-r)!}{(m-r)!} \sum_{s=0}^{n+\min(-m,r)} \frac{(m-n)_s(-n-r)_s}{s!(m-r+1)_s} + \mathrm{LDT} \\
&= \varepsilon^{n-m} a_\pm^{r+m} b_\pm^{r-m} J_0^{n-\max(|r|,|m|)} \frac{(-1)^{n-m}(n-r)!}{(m-r)!} F(m-n, -n-r; m-r+1; 1) + \mathrm{LDT} \\
&= \varepsilon^{n-m} a_\pm^{r+m} b_\pm^{r-m} J_0^{n-\max(|r|,|m|)} \frac{(-1)^{n-m}(2n)!}{(n+m)!} + \mathrm{LDT}
\end{aligned}
$$

whilst for $m \leq r$

$$
(\mathrm{Ad}_{J_-})^{n-m}(a_+^{n+r} b_+^{n-r}) = \varepsilon^{n-m} a_\pm^{r+m} b_\pm^{r-m} J_0^{n-\max(|r|,|m|)} \frac{(-1)^{n-r}(2n)!}{(n+m)!} + \mathrm{LDT}
$$

Using (4.2) gives result. □

Corollary 10. *The hermitian conjugate is given by*

$$
(\psi_n^{(r,m)})^\dagger = (-1)^{(r+m)} \psi_n^{(-r,-m)} \tag{5.16}
$$

Proof. By putting $m = -n$ in above we have the exact formula

$$
\psi_n^{(r,-n)} = (-1)^{r+n} a_-^{n-r} b_+^{n+r}
$$

Taking the conjugate of (4.2), using $(\mathrm{Ad}_{J_-}(f))^\dagger = -\mathrm{Ad}_{J_+}(f^\dagger)$ and substituting (5.17) gives (5.16). □

Theorem 11. *We can write the polynomials $\psi_n^{(r,m)}$ as*

$$
\psi_n^{(r,m)} = \begin{cases}
(-1)^{n-r} \varepsilon^{n-r} \binom{2n}{n+m}^{1/2} \binom{2n}{n+r}^{-1} a_+^{r+m}\, b_+^{r-m}\, h_{n-r}^{(r-m,r+m)} \left(\frac{(J_0+\hat{R})}{\varepsilon} - \frac{1}{2}, \frac{2\hat{R}}{\varepsilon} \right) \\
\qquad \text{for } -r \leq m \leq r \\
(-1)^{n-m} \varepsilon^{n-m} \binom{2n}{n+m}^{-1/2} a_+^{r+m}\, b_-^{m-r}\, h_{n-m}^{(m-r,r+m)} \left(\frac{(J_0+\hat{R})}{\varepsilon} - \frac{1}{2}, \frac{2\hat{R}}{\varepsilon} + r - m \right) \\
\qquad \text{for } -m \leq r \leq m \\
(-1)^{n-r} \varepsilon^{n+m} \binom{2n}{n+m}^{-1/2} a_-^{-r-m}\, b_+^{r-m} \times \\
\qquad h_{n+m}^{(r-m,-r-m)} \left(\frac{(J_0+\hat{R})}{\varepsilon} - \frac{1}{2} + r + m, \frac{2\hat{R}}{\varepsilon} + r + m \right) \qquad \text{for } m \leq r \leq -m \\
(-1)^{n-m} \varepsilon^{n+r} \binom{2n}{n+m}^{1/2} \binom{2n}{n+r}^{-1} a_-^{-r-m}\, b_-^{m-r} \times \\
\qquad h_{n+r}^{(m-r,-r-m)} \left(\frac{(J_0+\hat{R})}{\varepsilon} - \frac{1}{2} + r + m, \frac{2\hat{R}}{\varepsilon} + 2r \right) \qquad \text{for } r \leq m \leq -r
\end{cases}
$$

Proof. It is clear we shall have to consider the four cases separately. For each case the proof is not too different from [1, Theorem 7]. Let us here prove the theorem for the first case where $-r \le m \le r$, the other cases are Similarly proved.

Given two basis elements $\psi_{n_1}^{(r_1,m_1)}$ and $\psi_{n_2}^{(r_2,m_2)}$ it is clear that $\langle \psi_{n_1}^{(r_1,m_1)}, \psi_{n_2}^{(r_2,m_2)} \rangle = 0$ unless $m_1 = m_2$ and $r_1 = r_2$. Use lemma (2) so the $\psi_{n_1}^{(r,m)} = a_+^{r+m} b_+^{r-m} p(J_0, K_0)$. Thus $\psi_{n_1}^{(r,m)} = \rho(\psi_{n_1}^{(r,m)}) = a_+^{r+m} b_+^{r-m} p_{n_1}(J_0)$ for the polynomial $p_{n_1}(J_0)$ which we want to determine.

Let $R^2 = \varepsilon^2 k(k^1)$ so $R' = \varepsilon(k + \frac{1}{2})$ Then using (5.6) we have for $n_1 \ne n_2$

$$\begin{aligned}
0 &= \langle \psi_{n_1}^{(r,m)}, \psi_{n_2}^{(r,m)} \rangle \\
&= \sum_{j=-k}^{k} \langle k, j | p_{n_1}(J_0) a_+^{r+m} b_+^{r-m} a_+^{r+m} b_+^{r-m} p_{n_2}(J_0) | k, j \rangle \\
&= \sum_{j=-k}^{k} p_{n_1}(\varepsilon j) p_{n_2}(\varepsilon j) \frac{(k+j+r+m)!(k-j+r-m)!}{(k+j)!(k-j)!}
\end{aligned}$$

Thus $p_n(\varepsilon j)$ are polynomials orthogonal with respect to a weight function. This weight function is precisely the same as that for the Hahn polynomials $h_{n'}^{(\alpha,\beta)}(x, N)$ where $\alpha = r-m$, $\beta = r+m$, $x = j+k$ and $N = 2k+1$. We see that $n' = n-r$ since $h_{n'}^{(\alpha,\beta)}(x, N)$ is a polynomial of order n' and in x, and from (5.13) $p_n(J_0)$ is a polynomial of order $n-r$. Comparing the coefficient for the J_0^{n-r} term we have from [12]

$$h_{n-r}^{(r-m,r+m)} \left(\frac{(J_0 + \hat{R})}{\varepsilon} - \frac{1}{2}, \frac{2\hat{R}}{\varepsilon} \right) = \frac{(2n)!}{(n-r)!(r+n)!} \frac{J_0^{n-r}}{\varepsilon^{n-r}} + \mathrm{LDT}$$

Hence result for the range $-r \le m \le r$. Since this is true for all k then using lemma 6 gives result for all ε, R. The other ranges are proved similarly. $\square$

In the limit $\varepsilon = 0$ the Hahn polynomials become Jacobi Polynomials $(P_n^{(\alpha,\beta)}(z))$. Thus we have

$$\begin{aligned}
\psi_n^{(r,m)}|_{\varepsilon=0} = (-1)^{n-\max(r,m)} (2R)^{n-\max(|r|,|m|)} \binom{2n}{n+m}^{1/2} \binom{2n}{n-\max(|r|,|m|)}^{-1} \times \\
a_\pm^{r+m} b_\pm^{r-m} \, P_{n-\max(|r|,|m|)}^{(|r-m|,|r+m|)} \left(\frac{J_0}{R} \right) \qquad (5.18)
\end{aligned}$$

5.5 Another useful theorem

Here we extend [1, Lemma 9] to give a formula which is useful for the definition of the exterior derivative.

Theorem 12. *For r_1, r_2, n let*

$$\begin{aligned}
&\omega_n^{r_1 r_2} : \Psi \mapsto \Psi \\
&\omega_n^{r_1 r_2}(\xi) = \sum_{m=-n}^{n} \rho(\psi_n^{(r_2,m)\dagger} \, \xi \, \psi_n^{(r_1,m)})
\end{aligned} \qquad (5.19)$$

Then $\omega_n^{r_1 r_2}$ commutes with Ad_{J_0}, Ad_{J_+}, Ad_{J_-}, Δ and satisfies

It effect on the basis elements is given by

$$\omega_{n_1}^{r_1 r_2}(\psi_n^{(r,m)}) = \omega_{n_1 n}^{r_1 r_2 r}\psi_n^{(r+r_1-r_2,m)} \tag{5.21}$$

where $\omega_{n_1 n}^{r_1 r_2 r} \in \mathbb{C}$. *For* $\xi = 1$

$$\omega_n^{r_1 r_2}(1) = \delta_{r_1 r_2}(2n+1)\|\psi_n^{(r_1,m)}\|^2 \tag{5.22}$$

Proof. The effect of Ad_{J_0}, Ad_{J_+}, Ad_{J_-}, and Ad_{K_0} come from direct calculation using (4.4), (4.8), (4.9) and (4.6) respectively. The effect of Δ comes from Ad_{J_+} and Ad_{J_-}. From this, it is clear that $\omega_{n_1}^{r_1 r_2}(\psi_n^{(r,m)})$ must be proportional to $\psi_n^{(r+r_1-r_2,m)}$ and that the proportionality constant must be independent of m since $\omega_{n_1}^{r_1 r_2}$ commutes with Ad_{J_+}.

Since $\omega_n^{r_1 r_2}(1)$ is proportional to $\psi_0^{(r_1-r_2,0)}$ it is clear that it vanishes unless $r_1 = r_2$. In this case taking π_0 in the expression give the result. □

6 Physical Interpretation

There are two ways of interpreting the algebra (Ψ, ρ) and the basis elements $\psi_n^{(r,m)}$. In subsection 6.1 we show that $\psi_n^{(r,m)}$ my be thought of as the deformed rotation matrices for $SU(2)$, whilst in subsections 6.2, 6.3 and 6.4 we use them to construct the analogue of scalar, vector and spinor fields.

6.1 Deformed rotation matrices

In the limit $\varepsilon = 0$ the elements of Ψ may be regarded as complex functions over S^3 or $SU(2)$. The product ρ becomes pointwise multiplication. The basis elements $\psi_n^{(r,m)}$ are proportional to the rotation matrix D^n_{mr}. Thus we may interpret $\psi_n^{(r,m)}$ as the deformed (or noncommutative) rotation matrices.

Just as in standard quantum mechanics the non commutativity in the quantum domain becomes the Poisson Bracket in the commutative case we can define a bracket (6.4) as the limit of the commutator. However, since for $\varepsilon \neq 0$, ρ is not associative, this bracket is neither a derivation nor satisfies the Jacobi identity. Therefore it is not a Poison Bracket. This is reflected in the exisitance of the number operator δ_N in (6.5). It would be nice if one could find a physical interpretation for this bracket.

Lemma 13. *In the limit* $\varepsilon = 0$ *the noncommutative, nonassociative algebra* (Ψ, ρ) *becomes the commutative associative algebra* $C_0(SU(2))$*. This is the algebra of complex valued polynomial functions on* $SU(2)$ *with pointwise multiplication.*

If we use the Euler angles (α, β, γ) *to parameterise* $SU(2)$ *then the generators of* Ψ *maybe written*

$$\begin{aligned} a_+ &= \sqrt{2R}\cos(\beta/2)e^{-i/2(\alpha+\gamma)} & b_+ &= -i\sqrt{2R}\sin(\beta/2)e^{i/2(\alpha-\gamma)} \\ a_- &= \sqrt{2R}\cos(\beta/2)e^{i/2(\alpha+\gamma)} & b_- &= i\sqrt{2R}\sin(\beta/2)e^{-i/2(\alpha-\gamma)} \end{aligned} \tag{6.1}$$

The basis elements of Ψ *are given by the rotation matrices*

$$\psi_n^{(r,m)}|_{\varepsilon=0} = (i)^{m-r}(-1)^{n-r}(2R)^n \binom{2n}{r}^{-1/2} D^n_{mr}(\alpha,\beta,\gamma) \tag{6.2}$$

where $D^n_{mr}(\alpha,\beta,\gamma)$ uses the notation in [13]. *The bilinear form on Ψ becomes a true inner product and is given by*

$$\langle\xi,\zeta\rangle = \frac{1}{8\pi^2}\int_0^{2\pi} d\alpha \int_0^{2\pi} d\gamma \int_0^{\pi} d\beta \sin(\beta)\,\overline{\xi}\zeta \tag{6.3}$$

There is a bracket defined by

$$\{\bullet,\bullet\} : C_0(SU(2))\times C_0(SU(2))\mapsto C_0(SU(2)), \qquad \{\xi,\zeta\} = \lim_{\varepsilon\to 0}\tfrac{1}{i\varepsilon}\rho([\xi,\zeta]) \tag{6.4}$$

which is given by

$$\{\xi,\zeta\} = \frac{1}{R\sin\beta}\left(\frac{\partial\xi}{\partial\alpha}\frac{\partial\zeta}{\partial\beta} - \frac{\partial\xi}{\partial\beta}\frac{\partial\zeta}{\partial\alpha}\right) + \left(\delta_N(\xi)\frac{\partial\zeta}{\partial\gamma} - \frac{\partial\xi}{\partial\gamma}\delta_N(\zeta)\right) + \frac{\cot\beta}{R}\left(\frac{\partial\xi}{\partial\beta}\frac{\partial\zeta}{\partial\gamma} - \frac{\partial\xi}{\partial\gamma}\frac{\partial\zeta}{\partial\beta}\right) \tag{6.5}$$

where δ_N is the number operator

$$\delta_N : C_0(SU(2))\mapsto C_0(SU(2))\,, \qquad \delta_N(\psi_n^{(r,m)}) = n\psi_n^{(r,m)} \tag{6.6}$$

Proof. By setting $\varepsilon = 0$ it is clear that the algebra is commutative and associative. The condition that $K_0 = R$ implies that the elements of Ψ are functions on S^3, or alternatively $SU(2)$. Assuming (6.1) then (6.2) follows from (5.18). The inner product (6.3) follows from the orthogonality of the rotation matrix elements.

In the limit $\varepsilon = 0$ the algebra $\mathcal{W}(0)$ is the algebra $C_0(\mathbb{R}^4)$ of complex valued polynomial functions over $\mathbb{R}^4$. The Posion Bracket corresponding to the limit of the commutator is given by

$$\{f,g\}_{\mathcal{W}} = \frac{\partial f}{\partial a_+}\frac{\partial g}{\partial a_-} - \frac{\partial f}{\partial a_-}\frac{\partial g}{\partial a_+} + \frac{\partial f}{\partial b_+}\frac{\partial g}{\partial b_-} - \frac{\partial f}{\partial b_-}\frac{\partial g}{\partial b_+}$$

Writing this with respect to the coordinates (R,α,β,γ) gives (6.5) but with δ_N replaced by $\partial/\partial R$. However, for every polynomial $w(a_+,a_-,b_+,b_-)$ not explicitly dependent on R, $\partial w/\partial R = \delta_N(w)$. Thus restriction of $\{\bullet,\bullet\}_{\mathcal{W}}$ to $C_0(SU(2))$ is valid. □

6.2 The exterior derivative and 1-forms

In standard geometry there are many equivalent definitions of a vector field. However not all of the can be extended to the noncommutative case at the same time. If ones take the definition of a vector field given that it must satisfies the Leibniz formula, and extends this definition to noncommutative geometry then then, at least for the matrix case, the vector fields are given by $X = \mathrm{Ad}_f$ for some $f\in\Psi_\bullet^{(0,\bullet)}$. This can then be used to give a definition of the exterior derivative and the exterior algebra [2].

Here we give another definition of covectors by use of the analogue of the follow definition of the exterior derivative of scalar fields:

$$df = \sum_i \frac{\partial f}{\partial x^i}dx^i \tag{6.7}$$

If we consider the noncommutative sphere as a three dimensional manifold with normalised basis coordinates $x^m = (2\hat{R}+\varepsilon)^{-1/2}\psi_1^{(0,m)}$ for $m\in\{-1,0,1\}$ we can define the basis 1-forms as

This enables us to define the *exterior derivative* on the space of functions as

$$d : \Psi_\bullet^{(0,\bullet)} \mapsto \Psi_\bullet^{(-1,\bullet)} \qquad df = (2\hat{R}+\varepsilon)^{-1}\omega_1^{01}(f) \qquad \forall f \in \Psi_\bullet^{(0,\bullet)} \tag{6.9}$$

Since $\omega_1^{01}(1) = 0$, we see that

$$df = \sum_{m=-1}^{1} (-1)^{m+1} dx^{-m} \mathrm{Ad}_{x^m} f \tag{6.10}$$

which is analogous to (6.7). This obeys the following:

Lemma 14. *The two definition of the 1-forms dx^m given by (6.8) and (6.9) agree. For the basis elements which are functions*

$$d\psi_n^{(0,m)} = 2n(\hat{R}+\varepsilon n/2)(\hat{R}+\varepsilon)^{-1}\psi_n^{(-1,m)} \tag{6.11}$$

And in the limit as $\varepsilon \to 0$

$$d(fg) = d(f)g + f(dg) + O(\varepsilon) \tag{6.12}$$

Proof. From there definition we see that

$$\begin{aligned} J_+ &= \psi_1^{(0,1)} & -\sqrt{2}J_0 &= \psi_1^{(0,0)} & -J_- &= \psi_1^{(0,-1)} \\ b_-^2 &= \psi_1^{(-1,1)} & \sqrt{2}a_-b_- &= \psi_1^{(-1,0)} & a_-^2 &= \psi_1^{(-1,-1)} \end{aligned}$$

By manipulations of J_+, J_- we can show that

$$[J_-, J_+^n] = -2\varepsilon J_+^{n-1}(nz + \varepsilon n(n-1)/2)$$

From theorem 12 we know that $\omega_1^{01}(\psi_n^{(-1,m)})$ is proportional to $\psi_n^{(0,m)}$ and its coefficient is independent of m. Therefore taking $m = n \geq 1$ we have

$$\begin{aligned} \omega_1^{01}(\psi_n^{(0,n)}) &= a_-^2 \mathrm{Ad}_{J_+}(J_+^n) + 2a_-b_-\mathrm{Ad}_{J_0}(J_+^n) + b_-^2\mathrm{Ad}_{J_-}(J_+^n) \\ &= 2n\varepsilon a_-b_-J_+^n - 2\varepsilon b_-^2 J_+^{n-1}(nz + \varepsilon n(n-1)/2) \end{aligned}$$

Now $b_-^2 J_+^{n-1} = a_+^{n-1}b_-^{n+1} = \psi_n^{(-1,n)}$ whilst

$$\begin{aligned} a_-b_-a_+^n b_-^n &= (J_0 + K_0 + \varepsilon/2)a_+^{n-1}b_-^{n+1} \\ &= a_+^{n-1}b_-^{n+1}(J_0 + K_0 + n\varepsilon - \varepsilon/2)) \end{aligned}$$

Hence (6.11). We show that the two definitions of dx^m are equivalent by substituting $n = 1$ into this formula. Finally we note that

$$\sum_{m=-1}^{1} (\psi_1^{(1,m)^\dagger} fg\psi_1^{(0,m)} - \psi_1^{(1,m)^\dagger} f\psi_1^{(0,m)} g - f\psi_1^{(1,m)^\dagger} g\psi_1^{(0,m)}) = [\psi_1^{(1,m)^\dagger}, f][g, \psi_1^{(0,m)}] = O(\varepsilon^2)$$

□

6.3 Vector fields

We can now define the vectors fields as the set $\Psi_\bullet^{(1,\bullet)}$. The action of a vector on a scalar is given by

$$X(f) = \rho((df)\,X) \tag{6.13}$$

By the same reasoning as (6.12), this is also only a derivation in the limit.

$$X(fg) = X(f)g + fX(g) + O(\varepsilon) \tag{6.14}$$

However if we define X_m to be the conjugate of dx^m, $X_m = (dx^m)^\dagger = (-1)^{m+1}(2\hat{R}+\varepsilon)^{-1/2}\psi_1^{(1,-m)}$ then from $\omega_1^{1,0}(1) = \omega_1^{0,-1}(1) = 0$ we have

$$\sum_{m=-1}^{1} x^m X_m = \sum_{m=-1}^{1} X_m x^m = 0 \tag{6.15}$$

identically for all ε and not just in the limit $\varepsilon = 0$.

We can now define a *metric*

$$g : \Psi_\bullet^{(1,\bullet)} \times \Psi_\bullet^{(1,\bullet)} \mapsto \Psi_\bullet^{(0,\bullet)} \qquad\qquad g(X,Y) = \rho(X^\dagger Y) \tag{6.16}$$

This can be extended for all spaces $\Psi_\bullet^{(r,\bullet)}$.

6.4 Spinor fields

It is natural now to define the set $\Psi_\bullet^{(1/2,\bullet)}$ as the space of spinor fields and its dual $\Psi_\bullet^{(-1/2,\bullet)}$. This means that a vector is the product of two spinors. We may also consider the space $\Psi_n^{(r,\bullet)} = \Psi_n^{(\bullet,\bullet)} \cap \Psi_\bullet^{(r,\bullet)}$ as a $2n+1$ dimensional representation of $su(2)$ given by the adjoint representation, $\mathrm{Ad}_u : \Psi_n^{(r,\bullet)} \mapsto \Psi_n^{(r,\bullet)}$ for any non zero element $u \in su(2)$. The $2n+1$ eigenvalues of this mapping are given by the set $\{m\|u\|/2 \mid m+n \in \mathbb{Z}, |m| \leq n$, where $\|u\|$ corresponds to the killing form. Thus

$$e^{\pi i u/\|u\|}\psi_n^{(r,m)} e^{-\pi i u/\|u\|} = (-1)^{2n}\psi_n^{(r,m)}$$

Hence rotation by 2π does not change the sign of $\psi_n^{(r,m)}$ if n is an integer. i.e. for scalars, vectors and other "Bosons". Whilst $\psi_n^{(r,m)}$ changes sign under rotation of 2π for spinors tensor fields and other "Fermions".

An alternative way to define spinor fields is as the set

$$\mathcal{S} = \left\{ \begin{pmatrix} a_+ \\ b_+ \end{pmatrix} f_1 + \begin{pmatrix} a_- \\ b_- \end{pmatrix} f_2 \;\middle|\; f_1, f_2 \in \Psi_\bullet^{(0,\bullet)} \right\} \tag{6.17}$$

is usually regarded as the space of spinors. This is decomposed into $\mathcal{S} = \mathcal{S}_+ \oplus \mathcal{S}_-$ corresponding to the eigenspaces of Ad_{K_0} which is now regarded as the Chirality operator. This interpretation is now equivalent to the one proposed by Grosse et al [4, 9, 10], who go on to define and solve the Dirac equation. We note that $\mathcal{S}$ is not equivalent to simply two copies of $\Psi_\bullet^{(-1/2,\bullet)} \oplus \Psi_\bullet^{(1/2,\bullet)}$, but a proper subset. This is because the element $\begin{pmatrix} a_+ \\ 0 \end{pmatrix} \notin \mathcal{S}$. In the commutative limit $\varepsilon = 0$ we can replace $a_+ = a_- = R^{1/2}\cos(\theta/2)$ and $b_+ = \overline{b_-} = R^{1/2}\sin(\theta/2)e^{-i\phi}$ to obtain the standard

7 Problems and Outlook

There are many problems with this interpretation of our system. Here is a list of cases where this noncommutative geometry is different from the commutative geometry even in the limit $\varepsilon = 0$:

(1) the space $\Psi_\bullet^{(-1,\bullet)}$ is the image of $\Psi_\bullet^{(0,\bullet)}$ under d. This means that all 1-forms are closed.

(2) The definition (6.9) can be extended as a map $d : \Psi_\bullet^{(r,\bullet)} \mapsto \Psi_\bullet^{(r-1,\bullet)}$ for all r. However this map does not satisfy $d^2 = 0$ even in the limit. As a result no exterior calculus is defined here.

(3) The set of vectors X_i are not dual, in the usual sense, to the set of 1-forms dx^j, or alternatively the vectors X_i are not orthogonal with respect to the metric g. This is because for $i \neq j$

$$g(X_i, X_j) = \rho(dx^i X_j) = X_j(x^i) \neq 0$$

for all ε even when $\varepsilon = 0$. However we do have $\pi_0(g(X_i, X_j)) = \delta_{ij}$

Some of these problems, together with the non derivative nature of d may be solved by redefining the space of covectors. For example it may be similar to (6.17).

As stated at the end of subsection 6.1 the bracket (6.4) is in need of an interpretation.

Since ρ is not associative it would be nice to know what rules (if any) it obeys. Note, for example even $(\rho(\xi\xi)\xi) \neq \rho(\xi\xi\xi)$ so ξ^n is not well defined.

In the limit $\varepsilon = 0$ there is a formula for the combination of two matrix rotation entries D^n_{rm}. In the general case we can use the Wigner-Eckart theorem to give

$$\psi_{n_1}^{(r_1,m_1)} \psi_{n_2}^{(r_2,m_2)} = \sum_{n=|n_1-n_2|}^{n_1+n_2} C^{n_1,n_2,n}_{m_1,m_2,m_1+m_2} R^{n_1,n_2,n}_{r_1,r_2,r_1+r_2} \psi_n^{(r_1+r_2,m_1+m_2)}$$

where $C^{n_1,n_2,n}_{m_1,m_2,m_1+m_2}$ is the Wigner or Clebsch-Gordan coefficient and $R^{n_1,n_2,n}_{r_1,r_2,r_1+r_2}$ the corresponding reduced matrix element. It would be useful to have a formula for such elements. We note, however, that one cannot use the Wigner-Eckart theorem to reduce the problem on the r index since this is not raised and lowered by an action of $su(2)$.

Since the space of spinor and vector fields naturally form modules over the space of scalar fields one can use this as a stating point for algebraic connections [14].

As stated $\mathcal{W}$ contains a representation of $su(1,1)$. If one quotiented $\mathcal{W}$ by the ideals created by $J_0 \sim \hat{R}$, one would gain a very similar structure which could be interpreted as the basis of scalar, spinor and vector fields on the noncommutative analogue of either the two dimensional DeSitter space or the Hyperbolic plane.

The Jordan-Schwinger representation can be used for any Lie algebra. Hence one should be able to construct spinor fields on any symmetric manifold. The combination of these two steps would give one a description of scalar, spinor and vector fields on more exotic symmetric spaces, such as the Einstein-DeSitter Universe.

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References

[1] J. Gratus, *A Natural Basis of States for the Noncommutative Sphere and its Moyal bracket* to be published J. Math. Phys. (August 1997) q-alg/9703038

[2] J. Madore 1995, *An Introduction to Noncommutative Differential Geometry and its Physical Applications*, Cambridge University Press.

[3] U. Carow-Watamura, S. Watamura, *Chirality and Dirac Operators on the Noncommutative Sphere.* Comm. Math. Phys. **183**(2), 365-382 (1997)

[4] H. Grosse, C. Klimčík, P. Prešnajder, *Topologically Nontrivial Field Configurations in Noncommutative Geometry.* Comm. Math. Phys. **178**(2), 507-526 (1996)

[5] J. Madore, *The Commutative Limit of a Matrix Geometry.* J. Math. Phys. **32**(2), 332-335 (1991)

[6] H. Grosse, C. Klimčík, P. Prešnajder *Finite Quantum Field Theory in Noncommutative Geometry* Int. J. Theo. Phys. **35**(2), 231-244 (1996)

[7] J. Gratus, *Noncommutative Differential Geometry, and the matrix Representations of Generalised Algebras.* to be published J. Geometry Phys. (Accepted March 1997) q-alg/9703034

[8] H. Grosse, J. Madore, *A Noncommutative Version of the Schwinger Model* Phys. Lett. B **283** 218-222 (1992)

[9] H. Grosse, P. Prešnajder, *The Dirac Operator on the Fuzzy Sphere* Lett. Math. Phys. **33**(2) 171-181 (1995)

[10] H. Grosse, C. Klimčík, P. Prešnajder, *Field Theory on a Supersymmetric Lattice* Comm. Math. Phys. **185**, 155-175 (1997)

[11] M. Kibler, *On the* SU_2 *unit tensor* J. Math. Phys. **21**(3), 422-439 (1980)

[12] A. F. Nikiforov, S. K. Suslov, V. B. Uvarov 1991, *Classical Orthogonal Polynomials of a Discrete Variable,* Springer-Verlag

[13] L. C. Biedenharn J.D. Louck Angular momentum in Quantum Physics: Theory and Application Encyclopedia of Mathematics and its Aplications, Volume 8 Addison-Wesley 1981

[14] R. Coquereaux, *Algebraic Connections on Parallel Universes* Int. J. Mod. Phys. A **10**(1), 89-98 (1995)